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A Lower Bound on the Number of Cells in Arrangements of Hyperplanes*

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Lower bounds on the numbers of k-cells in arrangements of hyperplanes are found. These bounds are shown to be the best possible and arrangements attaining them are characterized.

1. INTRODUCTION

The main result in this paper is Theorem 4, which gives a lower bound on the number of k-cells in an arrangement of n hyperplanes in $\mathbf{P}^d$. For $k = 0$ this has already been established by Motzkin [6], Hanani [4], and Basterfield and Kelly [2]. The result is well-known for $d = 2$; for $k = d$ it was proved by McMullen in 1971 (unpublished), while Canham [3] proved the cases $k = d - 1$ and $k = d$. The general conjecture was first proposed by Grunbaum [7, p. 393]. To obtain Theorem 4 we utilize the presence of ordinary vertices in higher-dimensional arrangements. The existence of such vertices (or rather the dual result) was conjectured by Motzkin [6] and established by him for $d = 3$. Balomenos, Bonnice, and Silverman [1] proved Motzkin's conjecture for $d \leqslant 5$ and Hansen [5] established it for all d. We shall give a somewhat different formulation and simplified proof of Hansen's theorem.

After establishing the lower bounds for the numbers of k-cells in arrangements of n hyperplanes, we proceed to characterize the arrangements in which the lower bounds are obtained. This has been done (in the dual setting) for the case $k = 0$ by Basterfield and Kelly [2].

2. ARRANGEMENTS

A d-arrangement is a finite family $\mathscr{A}$ of hyperplanes in d-dimensional real projective or Euclidean space. If the members of $\mathscr{A}$ have nonempty

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intersection, then the arrangement is trivial. In this paper we will assume all arrangements to be nontrivial. For Euclidean d-arrangements we shall also require that each set of d hyperplanes have nonempty intersection. The k-flats of an arrangement $\mathscr{A}$ are the k-dimensional flats determined by the intersections of hyperplanes of $\mathscr{A}$. The hyperplanes of a d-arrangement partition $\mathbf{P}^d$ into a number of open regions whose closures are the d-cells of the arrangement. For $0 \leqslant k \leqslant d - 1$ the various k-faces of these cells are the k-cells of the arrangement.

A special type of arrangement important in the paper is the *near pencil*, a d-arrangement of n hyperplanes where $n - d + 1$ of the hyperplanes contain a $(d - 2)$-flat.

If $\mathscr{A}$ is a d-arrangement and M is a flat in $\mathbf{P}^d$, we define the induced arrangement $\mathscr{A}_M$ to be

$$\{M \cap H: H \in \mathscr{A}, M \not\subset H\}.$$

3. Ordinary Vertices

A vertex V of an arrangement $\mathscr{A}$ is called *ordinary* if there is a line l contained in all but one of the hyperplanes of through V. If L is the exceptional hyperplane, then the triple (V, l, L) is an *ordinary triple* for $\mathscr{A}$. Of course, if V is a simple vertex in a d-arrangement then there are d ordinary triples associated with it.

Theorem 1. *Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{E}^d$ where $n \geqslant d + 2$. Then there is an ordinary triple (V, l, L) for $\mathscr{A}$ and two vertices w_1 and w_2 of $\mathscr{A}$ which are contained in l and lie in opposite half-spaces determined by L.*

Proof. The proof is by induction on d. For $d = 1$ we have $n \geqslant 3$ and the conclusion of the theorem is true if we take $l = \mathbf{E}^1$, $w_1 = \min\{v \in \mathscr{A}\}$, $w_2 = \max\{v \in \mathscr{A}\}$, and $L = V$ to be any other point of $\mathscr{A}$. Now suppose that $d \geqslant 2$ and let $\mathscr{A}$ be an arrangement in $\mathbf{E}^d$. For each hyperplane H in $\mathscr{A}$ define a positive integer valued "height" function d_H on the vertices of $\mathscr{A}$ not incident with H in the following manner: Suppose v is in $\mathscr{A}$ but not in H. Each line l through v intersects H so that if p_1 and p_2 are two points on l then $d(p_1, H) \neq d(p_2, H)$ (where d is the Euclidean metric). We can define a broken path $\Sigma = \{[v_0, v_1], [v_1, v_2], \ldots, [v_{k-1}, v_k]\}$ where $v_0, \ldots, v_k$ are vertices of $\mathscr{A}$, $v_0 = v$, $v_k \in H$, $v_i \notin H$ for $1 \leqslant i \leqslant k - 1$, each segment $[v_i, v_{i+1}]$ for $0 \leqslant i \leqslant k - 1$ is an edge of $\mathscr{A}$, and $d(v_{i+1}, H) < d(v_i, H)$. Let $h(\mathscr{A}) = k$, the number of edges in Σ. Finally, set $d_H(v) = \max\{h(\Sigma)\colon \Sigma$ a broken path from v to $H\}$. It is clear that $d_H(v) \geqslant 1$ for each $H \in \mathscr{A}$ and $v \notin H$. We consider two cases:

(1) $d_H \equiv 1$ for each hyperplane H in $\mathscr{A}$. We show that this implies $n = d + 1$, contradicting the assumption that $n \geqslant d + 2$. If $d = 1$ then $d_H \equiv 1$ implies that $n = 2$. Suppose that $d \geqslant 2$ and let H be a hyperplane of $\mathscr{A}$. Since $d_{H'} \equiv 1$ for each hyperplane H' of $\mathscr{A}_H$ we can assume by induction that $\mathscr{A}_H$ is an arrangement of exactly d hyperplanes. Let $L_1, \ldots, L_d$ be the hyperplanes of $\mathscr{A}$ such that $\mathscr{A}_H = \{H \cap L_1, \ldots, H \cap L_d\}$. Let $p = \bigcap_{i=1}^{d} L_i$ and let $L \in \mathscr{A}$ intersect H in $L_1 \cap H$. Then $L \cap L_2$ is a hyperplane in $\mathscr{A}_{L_2}$ which contains all vertices of $\mathscr{A}_{L_2}$ except $L_2 \cap \cdots L_d \cap H$ and hence contains p. Since p and $L_1 \cap H$ together span both L and L_1 we must have $L_1 = L$. Hence $\mathscr{A} = \{L_1, \ldots, L_d, H\}$. |||

(2) There exists a hyperplane H and vertex v not in H such that $d_H(v) = k \geqslant 2$: Let $\Sigma = \{[v_0, v_1], [v_1, v_2], \ldots, [v_{k-2}, v_{k-1}], [v_{k-1}, v_k]\}$ be a path of maximal length from v to H. Then we must have $d_H(v_{k-1}) = 1$. Consider the arrangement $\mathscr{A}_{H,v_{k-1}} = \{H \cap K\colon v_{k-1} \in K, K \in \mathscr{A}\}$ induced in H by hyperplanes through v_{k-1}. We have two possibilities.

(i) Card $\mathscr{A}_{H,v_{k-1}} = d$: Then v_{k-1} is a simple vertex in $\mathscr{A}$. We can take $l =)v_{k-2}, v_{k-1}($, the line determined by v_{k-2} and v_{k-1}, $w_1 = v_{k-1}$ and $w_2 =)v_{k-2} \cdot v_{k-1}(\cap H)$.

(ii) Card $\mathscr{A}_{H,v_{k-1}} \geqslant d + 1 = (d - 1) + 2$: By the induction hypothesis there is an ordinary triple (V, l, L') for $\mathscr{A}_{H,v_{k-1}}$ together with two vertices, w_1 and w_2, on l which are separated in H by L'. Let L be the hyperplane of $\mathscr{A}$ containing v_{k-1} which gives $L' = H \cap L$. We claim that (V, l, L), w_1 and w_2 satisfy the conclusion of the theorem. To prove this, we must show that if K is a hyperplane of $\mathscr{A}$ with $V \in K$ and $l \not\subset K$, then $K = L$. We show first that $v_{k-1} \in K$. Suppose the contrary. Since $l \not\subset K$, K separates w_1 and w_2, and because $v_{k-1} \notin K$ we have v_{k-1} and one of $\{w_1, w_2\}$, say w_1, lying in the two opposite halfs-paces determined by K. Hence $K \cap [v_{k-1}, w_1]$ is a vertex of $\mathscr{A}$ and it follows that $d_H(v_{k-1}) > 1$, contrary to our assumption concerning the maximality of Σ. Therefore $v_{k-1} \in K$ and $H \cap K \in \mathscr{A}_{H,v_{k-1}}$. Since $l \not\subset H \cap K$ and V is ordinary in $\mathscr{A}_{H,v_{k-1}}$ we have $H \cap K = H \cap L$ and hence $K = L$. This completes the induction and the proof of the theorem.

The following corollary will allow us to proceed by induction in many of the proofs that follow in the next two sections.

Corollary 1. *Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{P}^d$ where $d \geqslant 1$ and $n \geqslant d + 2$. Then there exists an ordinary triple (V, l, L) for $\mathscr{A}$*

such that $\mathscr{A} - \{L\}$ is an arrangement. That is, the hyperplanes of $\mathscr{A}$ different from L do not all pass through the same point.

Proof. The proof proceeds by induction on d. For $d = 1$ the corollary is certainly true since in this case $n \geqslant 3$. Suppose that $d \geqslant 2$ and let $\mathscr{A}$ be an arrangement in $\mathbf{P}^d$. If $\mathscr{A} - \{L\}$ is nontrivial for each hyperplane L of $\mathscr{A}$ then the corollary follows from the existence of ordinary vertices in $\mathscr{A}$. Hence we may suppose there exists a hyperplane L_0 in $\mathscr{A}$ such that $\mathscr{A} - \{L_0\}$ is trivial. Consider the arrangement $\mathscr{A}_{L_0}$ induced in L_0. By the induction hypothesis $\mathscr{A}_{L_0}$ has an ordinary triple (V, l, L') (where $L' = L_0 \cap L$ for some L in $\mathscr{A}$) such that $\mathscr{A}_{L_0} - \{L'\}$ is nontrivial. It follows that $\mathscr{A} - \{L\}$ is nontrivial and it is easy to see that (V, l, L) is an ordinary triple for $\mathscr{A}$. ||

Henceforth an ordinary triple (V, l, L) for which $\mathscr{A} - \{L\}$ is an arrangement will be called a *good triple.*

4. k-Flats

Before proceeding to the problem of k-cells, we determine lower bounds on the numbers of k-flats in an arrangement.

For $0 \leqslant K \leqslant d - 1$ let $g_k(\mathscr{A})$ be the number of k-dimensional flats in an arrangement $\mathscr{A}$ in $\mathbf{P}^d$. Let

$$\psi_k{}^d(n) = \binom{d+1}{k+1} + (n - d - 1)\binom{d-1}{k}.$$

Theorem 2. *Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{P}^d$. Then*

$$g_k(\mathscr{A}) \geqslant \psi_k{}^d(n) \qquad 0 \leqslant k \leqslant d - 1$$

with equality holding for near pencils.

Proof. The proof is by induction on n. The case $n = d + 1$ is easily computed. Suppose that $n \geqslant d + 2$ and let (V, l, L) be a good triple of $\mathscr{A}$. Let $\Omega = \{H_1, ..., H_{d-1}\}$ be a family of hyperplanes through V such that $\bigcap \Omega = l$. Let $\Gamma \subset \Omega$ be a subset of $d - k - 1$ $(0 \leqslant k \leqslant d - 1)$ hyperplanes. Then $\bigcap \Gamma \cap L$ is a k-dimensional flat of $\mathscr{A}$. If $\Gamma_1 \subset \Omega$, $\Gamma_1 \neq \Gamma$ then $\bigcap \Gamma_1 \cap L \neq \bigcap \Gamma \cap L$. Suppose Λ is a family of hyperplanes of $\mathscr{A} - \{L\}$ with $\bigcap \Lambda \supset \bigcap \Gamma \cap L$. Then $V \in \bigcap \Lambda$ and hence $l \subset \bigcap \Lambda$, so that

$\bigcap \Lambda \neq \bigcap \Gamma \cap L$. There are $\binom{d-1}{d-k-1}$ different ways to choose Γ. Hence

$$\begin{aligned} g_k(\mathscr{A}) &\geqslant g_k(\mathscr{A} - \{L\}) + \binom{d-1}{d-k-1} \\ &\geqslant \binom{d+1}{k+1} + ((n-1) - d - 1)\binom{d-1}{k} + \binom{d-1}{k} \\ &= \binom{d+1}{k+1} + (n-d-1)\binom{d-1}{k}. \end{aligned}$$

Since a near pencil of n hyperplanes in $\mathbf{P}^d$ is the join of a near pencil of $n-1$ hyperplanes in $\mathbf{P}^{d-1}$ with a point, the statement concerning equality also holds. |||

For $k = 0$ we will see in Theorem 8 that equality can hold in Theorem 2 for a larger class of arrangements than the near pencils. However, for $0 < k < d - 2$ it may be conjectured that equality holds only for near pencils.

5. k-Cells

We now turn to the problem of determining the minimal numbers of k-cells in arrangements.

Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{P}^d$. For $0 \leqslant k \leqslant d$ let $f_k(\mathscr{A})$ be the number of k-dimensional cells of $\mathscr{A}$ and let

$$\phi_k{}^d(n) = 2^k \binom{d}{k} n + 2^{k-1}\binom{d-1}{k-1}(d+1-n) - 2^k\binom{d+1}{k+1} k.$$

We shall prove that $f_k(\mathscr{A}) \geqslant \phi_k{}^d(n)$ and characterize those arrangements where equality holds. First we prove a result of McMullen (see [3]).

THEOREM 3. *Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{P}^d$. Then*

$$f_d(\mathscr{A}) \geqslant \phi_d{}^d(n) = 2^{d-1}(n-d+1).$$

Proof. The proof is by induction on n. The theorem is true for $n = d + 1$. Suppose $n \geqslant d + 2$ and let H be any hyperplane of $\mathscr{A}$ such that $\mathscr{A} - \{H\}$ is nontrivial. Such an H is given by Corollary 1. Let $\mathscr{A}_H$ be the standard induced arrangement in H. Then since $\mathscr{A}_H$ is nontrivial it has at least 2^{d-1} $(d-1)$-dimensional cells. Each of these cells divides a d-cell of $\mathscr{A}$ into two d-cells. Hence we have

$$\begin{aligned} f_d(\mathscr{A}) &\geqslant f_d(\mathscr{A} - \{H\}) + 2^{d-1} \\ &\geqslant 2^{d-1}((n-1) - d + 1) + 2^{d-1} \\ &= 2^{d-1}(n-d+1) \\ &= \phi_d{}^d(n). \quad ||| \end{aligned}$$

THEOREM 4. *Let $\mathscr{A}$ be an arrangement of n hyperplanes in* $\mathbf{P}^d$. *Then*

$$f_k(\mathscr{A}) \geqslant \phi_k{}^d(n) \qquad 0 \leqslant k \leqslant d-1.$$

Proof. The proof is by induction on n. The inequality can easily be checked for $n = d + 1$. Let $\mathscr{A}$ be an arrangement of n hyperplanes in $\mathbf{P}^d$ with $n \geqslant d + 2$ and let (V, l, L) be a good triple in $\mathscr{A}$. Let Ω be a family of $d - 1$ hyperplanes through V such that $\bigcap \Omega = l$. Let $\Gamma \subset \Omega$ be a subset of $k - 1$ hyperplanes. Then $\bigcap \Gamma \cap L$ is a $(d - k)$-dimensional flat. As was seen in the proof of Theorem 2, if Θ is a family of hyperplanes of $\mathscr{A}$ not including L then $\bigcap \Theta \neq \bigcap \Gamma \cap L$. The arrangement $\mathscr{A}_{\cap \Gamma \cap L}$ induced in $\bigcap \Gamma \cap L$ contains at least $2^{d-k}(d - k)$-cells, the number of $(d - k)$-cells in a $(d - k)$-dimensional extended simplex. There are $\binom{d-1}{k-1}$ ways to choose Γ so we get at least $\binom{d-1}{k-1} 2^{d-k}(d - k)$-cells in this manner. Note that if $\Gamma_1 \subset \Omega$, $\Gamma_2 \subset \Omega$ and $\Gamma_1 \neq \Gamma_2$ then $\bigcap \Gamma_1 \cap L$ and $\bigcap \Gamma_2 \cap L$ intersect in at most a $(d - k - 1)$-dimensional flat, so that no $(d - k)$-cells are counted more than once. Now suppose $\Lambda \subset \Omega$ contains k hyperplanes. Then $\bigcap \Lambda$ is a $(d - k)$-flat of $\mathbf{P}^d$. Let $\bigcap \Lambda \cap L = R$ be the $(d - k - 1)$-dimensional flat induced in $\bigcap \Lambda$ by L. If there is another hyperplane H in $\mathscr{A}$ with $\bigcap \Lambda \cap H = R$, then $V \in H$, and since V is ordinary, $l \subset H$. But then $l \subset R$, which is impossible since $R = \bigcap \Lambda \cap L$ and $l \not\subset L$. Hence L is the only hyperplane in $\mathscr{A}$ to cut $\bigcap \Lambda$ in R. Let $\mathscr{A}_{\cap \Lambda} - \{R\}$ be the arrangement induced in $\bigcap \Lambda$ by the hyperplanes of $\mathscr{A}$ different from L. The addition of L to $\mathscr{A}$ adds at least $2^{d-k-1}(d - k)$-cells to $\mathscr{A}_{\cap \Lambda}$ and hnece to $\mathscr{A}$. There are $\binom{d-1}{k}$ ways to pick $\Lambda \subset \Omega$ and if $\Lambda_1 \neq \Lambda_2$ then $\bigcap \Lambda_1 \neq \bigcap \Lambda_2$. Hence we add at least $\binom{d-1}{k} 2^{d-k-1}$ $(d - k)$-cells in this manner, bringing the total surplus of $(d - k)$-cells in $\mathscr{A}$ over $\mathscr{A} - \{L\}$ to $\binom{d-1}{k-1} 2^{d-k} + \binom{d-1}{k} 2^{d-k-1}$. Hence we have

$$\begin{aligned}
f_{d-k}(\mathscr{A}) &\geqslant f_{d-k}(\mathscr{A} - \{L\}) + \left[\binom{d-1}{k-1} 2^{d-k} + \binom{d-1}{k} 2^{d-k-1}\right] \\
&\geqslant \phi_{d-k}^{d}(n-1) + \left[\binom{d-k}{k-1} 2^{d-k} + \binom{d-1}{k} 2^{d-k-1}\right] \\
&= 2^{d-k}\binom{d}{k}(n-1) + 2^{d-k-1}\binom{d-1}{k}(d+1-(n-1)) \\
&\quad - 2^{d-k}\binom{d+1}{d-k+1}(d-k) \\
&\quad + \left[2^{d-k}\binom{d}{k} - 2^{d-k-1}\binom{d-1}{k}\right] \\
&= \phi_{d-k}^{d}(n). \quad |||
\end{aligned}$$

In the next four theorems we shall characterize arrangements $\mathscr{A}^d(n)$ for which $f_k(\mathscr{A}^d(n)) = \phi_k{}^d(n)$ for some $0 \leqslant k \leqslant d$.

THEOREM 5. *Let $d \geqslant 2$. If $f_d(\mathscr{A}^d(n)) = \phi_d{}^d(n)$ then $\mathscr{A}^d(n)$ is a near pencil.*

Proof. The proof is by induction on d. If H is a hyperplane of $\mathscr{A}$ such that $\mathscr{A} - \{H\}$ is nontrivial then, by the proof of Theorem 2, $\mathscr{A}_H$ is an extended simplex. It must follow that for some H_0, $\mathscr{A} - \{H_0\}$ is trivial. For $d = 2$ this implies directly that $\mathscr{A}$ is a near pencil. Otherwise, $f_d(\mathscr{A}) = 2f_{d-k}(\mathscr{A}_{H_0})$ so that $f_{d-1}(\mathscr{A}_{H_0}) = \frac{1}{2}(\phi_d{}^d(n)) = \phi_{d-1}^{d-1}(n-1)$ which forces $\mathscr{A}_{H_0}$ to be a near pencil. Hence $\mathscr{A}$ is a near pencil. |||

THEOREM 6. *Let $d \geqslant 2$. If $f_{d-1}(\mathscr{A}^d(n)) = \phi_{d-1}^d(n)$ then $\mathscr{A}^d(n)$ is a near pencil.*

Proof. The proof proceeds by induction on n. The theorem is certainly true for $n = d + 1$. Suppose $n \geqslant d + 2$ and let H be the hyperplane of a good triple. Let $\bar{\mathscr{A}} = \mathscr{A} - \{H\}$. Then the proof of Theorem 4 shows that $f_{d-1}(\bar{\mathscr{A}}) = \phi_{d-1}^d(n-1)$, (and hence that $\bar{\mathscr{A}}$ is a near pencil), and also that $\mathscr{A}_H$ is an extended simplex.

Case 1. $n - 1 = d + 1$: Then d of the hyperplanes of $\bar{\mathscr{A}}$ intersect H in an extended simplex and the last must also meet H at one of these induced $(d-2)$-flats. It follows that $\mathscr{A}^d(n)$ is a near pencil.

Case 2. $n - 1 \geqslant d + 2$: Then $(n-1) - d + 1$ of the hyperplanes of $\bar{\mathscr{A}}$ intersect in a $(d-2)$-dimensional flat F. We show that $F \subset H$: Suppose $L_1, \ldots, L_{d-1}$ are the hyperplanes of $\bar{\mathscr{A}}$ which do not contain F. Then $\bar{\mathscr{A}}_F = \{L_1 \cap F, \ldots, L_{d-1} \cap F\}$ is an extended simplex in F and hence its vertices span F. We show that each vertex of $\bar{\mathscr{A}}_F$ is in H. Any such vertex v is contained in $(d-2) + (n-1) - d + 1) = n - 2 \geqslant d + 1$ hyperplanes of $\bar{\mathscr{A}}_F$. If H misses v then there must be at least $d + 1$ hyperplanes in $\mathscr{A}_H$, contradicting the fact that $\mathscr{A}_H$ is an extended simplex and has exactly d hyperplanes. Hence $v \in H$ for each $v \in \mathscr{A}_F$ and so $F \subset H$. Thus $A^d(n)$ is a near pencil. |||

THEOREM 7. *Suppose that $2d \leqslant k \leqslant d - 1$ and $f_{d-k}(\mathscr{A}^d(n)) = \phi_{d-k}^d(n)$. Then $\mathscr{A}^d(n)$ is a near pencil.*

Proof. We assume $n \geqslant d + 2$. Let (V, l, L) be a good triple in $\mathscr{A}$. We show that there is a $(d-2)$-dimensional flat K contained in all but $d - 1$ of the hyperplanes of $\mathscr{A}$.

Claim. If $H_1, \ldots, H_p$ are the hyperplanes containing l, then $p = d - 1$:

If $H_r \cap L = H_s \cap L$ then $H_r \cap H_s$. Hence, if M is any hyperplane in $\mathbf{P}^d$ not containing V then there are p distinct hyperplanes in the arrangement $\mathscr{A}_{M \cap L, V} = \{H_s \cap L \cap M : s = 1,..., p\}$. The number of distinct $(d-k)$ flats through V in L is the number of distinct $(d-k-1)$ flats in $\mathscr{A}_{M \cap L, V}$ which is, by Theorem 2, at least

$$\psi^{d-2}_{d-k-1}(p) = \binom{d-1}{d-k} + (p-d+1)\binom{d-3}{d-k-1} > \binom{d-1}{d-k}$$
$$= \binom{d-1}{k-1} \quad \text{if} \quad p > d-1.$$

But the proof of Theorem 4 shows that there must not be more than $\binom{d-1}{d-k}$ distinct $(d-k)$-flats through V in L, thus proving the claim. Now the proof of Theorem 4 also implies that $\mathscr{A}_{\bigcap \mathscr{B} \cap L}$ is an extended simplex for each $\mathscr{B} \subset \{H_1, ..., H_{d-1}\}$ containing $k-1$ hyperplanes. Note that $\dim(\bigcap \mathscr{B} \cap L) = d-k \geqslant d-(d-1) = 1$. Hence for each $s (1 \leqslant s \leqslant d-1)$, the line $\bigcap_{r \neq s} H_r \cap L$ contains exactly two vertices of $\mathscr{A}^d(n)$, one of which we know to be V. Call the other V_s. Let $K =$ projective hull $\{V_1, ..., V_{d-1}\}$. Then any hyperplane of $\mathscr{A}^d(n)$ different from $H_1, ..., H_{d-1}$ contains V_s $1 \leqslant s \leqslant d-1$ and thus contains K. |||

The following theorem has been proven in the dual formulation by Basterfield and Kelly [2]. We begin with a short lemma.

LEMMA 1. *Let $\mathscr{A}$ be an arrangement in $\mathbf{P}^d$ such that all vertices of $\mathscr{A}$ but one, say p, lie in a hyperplane $H \subset \mathbf{P}^d$. Then H is a hyperplane of $\mathscr{A}$ and $\mathscr{A} - \{H\}$ is trivial.*

Proof. If $L \neq H$ is a hyperplane of $\mathscr{A}$ then $p \in L$: There is a vertex of $\mathscr{A}$ contained in $L \backslash H$ (the vertices of $\mathscr{A}$ in L must span L) and hence it must be p. If $H \in \mathscr{A}$ then $\mathscr{A}$ would itself be trivial. |||

THEOREM 8. *Let $\mathscr{A}$ be an arrangement in $\mathbf{P}^d$ with n hyperplanes and n vertices. Then there is a line l and $(d-2)$ flat K with $l \cap K = \varnothing$ such that $K \subset H$ or $l \subset H$ for each hyperplane H in $\mathscr{A}$. If r of the hyperplanes of $\mathscr{A}$ contain l then $\mathscr{A}_K$, the induced arrangement in K, has r vertices.*

Proof. The proof is by induction on n. The theorem is certainly true for $n = d+1$ so let $n \geqslant d+2$. Let H be a hyperplane in $\mathscr{A}^d(n)$ as given by Corollary 1. Then $\bar{\mathscr{A}}(n-1) = \mathscr{A}^d(n) - H$ has $n-1$ vertices and so there exists a line l and suitable $(d-2)$ flat K satisfying the conclusions of the theorem. If H contains K or l we are done. Otherwise H intersects K in a $(d-3)$ flat K and l in a point p. Consider the arrangement $\mathscr{A}_H$

induced in H: $\mathscr{A}_H$ has at least $n - 2$ hyperplanes: For if L_1 and L_2 both contain l (or K), then $H \cap L_1 = H \cap L_2$ implies $L_1 = L_2$. Hence if $H \cap L_1 = H \cap L_2$ but $L_1 \neq L_2$ then $K \subset L_1$ and $l \subset L_2$. This means, since $p \in H \cap L_2$, that $p \in L_1$. If $H \cap L_3 = H \cap L_4$ with $K \subset L_3$, $l \subset L_4$ then similarly $p \in L_3$; hence $L_3 = L_1$ and $L_4 = L_2$. This proves the claim. By Theorem 4 it follows that $\mathscr{A}_H$ has at least $n - 2$ vertices and $n - 3$ of these must already be vertices of $\bar{\mathscr{A}}(n - 1)$. Hence, since we are assuming H does not contain K or l, $\bar{K} = H \cap K$ contains all but one, say p_1, of the vertices of $\mathscr{A}(n) \cap K$ and there are just 2 vertices of $\mathscr{A}(n)$ in l, one of which is p. Call the other p_2. Let $K^* =$ projective hull $(\bar{K} \cup \{p\})$ and $l^* =)p_1, p_2($. Since $p \notin K$, $\dim K^* = d - 2$. To complete the proof we must show that if $L \in \mathscr{A}(n)$ and $l^* \not\subset L$, then $K^* \subset L$. So suppose $l^* \not\subset L \neq H$. Then either $p_1 \notin L$ or $p_2 \notin L$. If $p_2 \notin L$ then $l \not\subset L$ so $K \subset L$ and since $L \cap l \neq \varnothing$, $p \in L$. Hence $K^* \subset L$. Suppose on the other hand that $p_1 \notin L$. Then $K \not\subset L$ since $p_1 \in K$. Therefore $l \subset L$ and thus $p \in L$. The fact that $\bar{K} \subset L$ follows from the lemma. |||

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